4 Mathematically speaking

Statement of inquiry:

Understanding health and validating life-style choices results from using logical representations and systems.

Key concept:

Logic is a method of reasoning and a system of principles used to build arguments and reach conclusions.

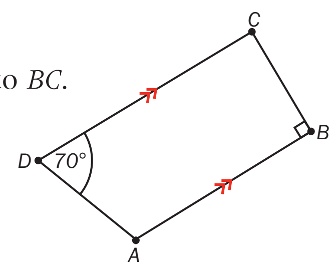
F How can representing language with symbols facilitate the operations?

Is a picture worth a thousand words?

ABCD is a quadrilateral. *AB* is parallel to *CD* and perpendicular to *BC*.

Angle $ADC = 70°$. Find angle *DAB*.

- How does drawing a diagram help you solve this problem?
- Could you solve it without a diagram?



C How do axioms enhance the understanding of logic?

Axiomatic system for number

You have been working with the real number system since you first learned to count. The number system is based on a set of axioms, which are assumed to be true for real numbers.

Verify the axioms below for real numbers x, y and z.

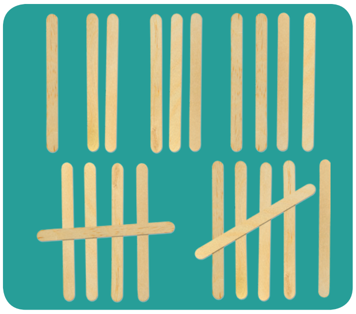
Addition and multiplication are:

Commutative: $x + y = y + x$
$xy = yx$

Associative: $(x + y) + z = x + (y + z)$
$(xy)z = x(yz)$

Distributive: $x(y + z) = xy + xz$

These axioms form the basis for all of your number skills.



D What factors validate our life-style decisions?

To validate is to use well-founded, logical mathematics to come to a true and accurate conclusion or a reasonable interpretation of results.

Justification in medical research

Pharmaceutical companies are continually trying to produce new medicines to treat as-yet incurable conditions, or to treat curable conditions more effectively. Before a new drug can go on sale, the company needs to justify that the drug is safe and effective. To do this, they run clinical trials to test the drugs on humans and collect evidence on positive results as well as negative side effects.

All clinical trials have to be approved by a scientific and ethical committee. The researchers must justify that the volunteers in the trial will not be exposed to unnecessary risk, and that the new drug can reasonably be expected to improve patient care.

Ecosystems

When an ecosystem is in balance, all the living organisms within it are healthy and capable of reproducing themselves. If one part of the ecosystem is damaged – either by natural events or human activity – every other element of the ecosystem is affected.

Forest fires cause widespread destruction, but fire is a natural and vital element in some forest ecosystems. After a fire, more sunlight can reach the forest floor, which is free from dead wood and leaf litter and has increased nutrient levels from the ash. This creates perfect conditions for some trees and plants to thrive. Some species have evolved to withstand fire (or sometimes even rely on it) as part of their life cycle. For example, sand pine cones open to disperse their seeds only in intense heat.

Global context: Identities and relationships

Exploration: Explore personal and physical health and good lifestyle choices

Launch additional digital resources for this chapter.

E4.1 Independent events and conditional probability

Global context: Identities and relationships

Related concept: Systems

LOGIC

Objectives

- Drawing diagrams to represent and calculate independent and dependent events
- Determining whether two events are independent or dependent events

Inquiry questions

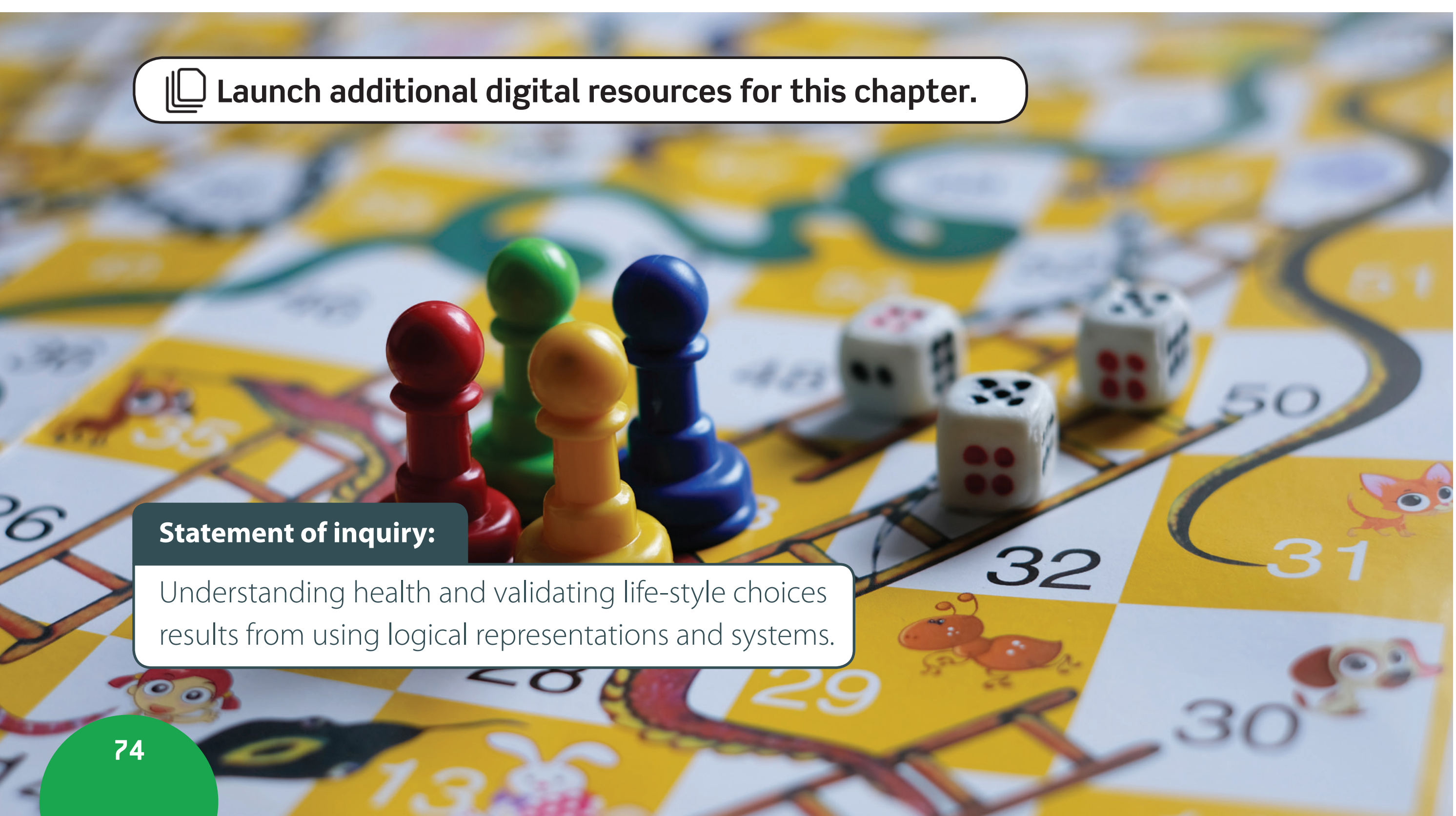
F
- What is the addition rule?
- What are independent events?
- What is conditional probability?

C
- How can an axiomatic system be developed?
- How does a system facilitate the solving of problems?

D
- What factors affect the decisions we make?

ATL Communication

Use and interpret a range of subject specific terms and symbols

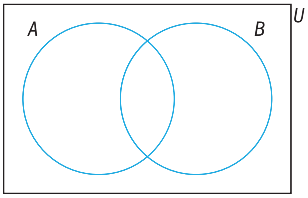


Launch additional digital resources for this chapter.

Statement of inquiry:

Understanding health and validating life-style choices results from using logical representations and systems.

You should already know how to:

• draw tree diagrams	**1** Caesar's new playlist contains ten songs from the 1980s and eight songs from the 1990s. He selects two songs at random, without choosing the same song twice. **a** Draw a tree diagram to represent this situation. **b** Find the probability that the two songs are both from the 1980s.
• use Theorem 1 (Addition rule) and Theorem 2 (Multiplication rule)	**2** You have a standard deck of 52 playing cards. **a** You take one card. What is the probability that you take either a red card or a face card? **b** You take one card, replace it, and then take another. What is the probability that you take a 5 followed by a face card? **c** You take one card and then take another, without replacement. What is the probability that you take a 5 followed by a face card?
• calculate probabilities of mutually exclusive events	**3** A box contains 12 soft-center chocolates and 15 hard-center chocolates. One chocolate is picked at random from the box. Let A be the event 'pick a hard center' and let B be the event 'pick a soft center'. **a** Find P(A). **b** Find P($A \cup B$).
• use Venn diagrams and set notation	**4** There are 33 students taking Art and 28 students taking Biology. There are 7 students taking both subjects. Represent this with a Venn diagram. **5** Make a copy of the Venn diagram below for each part of this question.  Shade the part of the Venn diagram that represents: **a** B **b** B' **c** $A \cap B$ **d** $A \cup B'$ **e** $(A \cap B)'$

F Probabilities of combined events

- What is the addition rule?
- What are independent events?
- What is conditional probability?

ATL

Exploration 1

1 Construct a Venn diagram to illustrate this information:

- 30 students study one or more of three languages: French, German and Mandarin
- 5 study all three languages
- 6 study Mandarin only
- 2 study French and Mandarin but not German
- 15 students in total study Mandarin
- 2 study only French
- 3 study only German.

2 From your diagram, find n(French) and $n(U)$. Use these to find the probability that a student picked at random studies French.

3 Find n(French and Mandarin) and P(French and Mandarin) from your diagram. State what each of these numbers represent.

4 Explain how you can find P(French ∩ German ∩ Mandarin) from your Venn diagram.

5 Summarize your findings:

$P(F) =$	$P(F \cap G) =$
$P(G) =$	$P(G \cap M) =$
$P(M) =$	$P(F \cap G \cap M) =$
$P(F \cap M) =$	

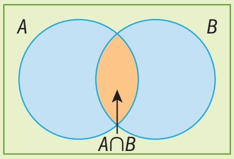
6 Find n(French ∪ Mandarin) and P(French ∪ Mandarin) from your Venn diagram. Use the probabilities you found in step **5** to find and verify a formula for P(French ∪ Mandarin). Do the same for P(French ∪ German) and P(Mandarin ∪ German).

Your results from Exploration 1 lead to the addition rule, used to determine the probability that event A **or** event B occurs, **or** both occur.

Theorem 1: Addition rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Calculating $P(A) + P(B)$ adds the intersection $P(A \cap B)$ twice, so you need to subtract it once.



A fair die is rolled once.

Let A be the event 'rolling a 1'.

Let B be the event 'rolling a 2', and so on.

$\mathrm{P}(A)=\frac{1}{6}$ $\mathrm{P}(B)=\frac{1}{6}$ $\mathrm{P}(C)=\frac{1}{6}$ $\mathrm{P}(D)=\frac{1}{6}$ $\mathrm{P}(E)=\frac{1}{6}$ $\mathrm{P}(F)=\frac{1}{6}$

The events are mutually exclusive as it is impossible, for example, to roll a 1 and a 2 at the same time, so $\mathrm{P}(A \cap B) = 0$.

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$
$$= \frac{1}{6} + \frac{1}{6} - 0 = \frac{2}{6}$$

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B \cap C)$$
$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} - 0 = \frac{3}{6}$$

Italian mathematician Gerolamo Cardano (1501–76) analysed the likelihood of events with both cards and dice. His *Liber de Ludo Aleae* (Book on Games of Chance) discussed in detail many of the basic concepts of probability theory, but received little attention. Cardano often found himself short of money, so he would gamble as a means to make some. His *Ludo Aleae* even contains tips on how to cheat by using false dice and marked cards!

Independent events

If you roll a die twice, the outcome of the first roll does not affect the probabilities of the outcomes of the second roll. The results of the two rolls are independent of each other.

If the outcome of an event in one experiment does not affect the probability of the outcome of the event in the second experiment, then the events are **independent**.

Theorem 2: Multiplication rule

If A and B are independent events, then $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$.

ATL

Example 1

Tickets numbered 1, 2, 3, 4, 5, 6, 7, 8 and 9 are placed in a bag.

One ticket is then taken out of the bag at random.

Let A be the event 'the ticket's number is even', and let B be the event 'the ticket's number is a square number'.

a Represent the information on a Venn diagram.

b Calculate $\mathrm{P}(A)$, $\mathrm{P}(B)$ and $\mathrm{P}(A \cap B)$.

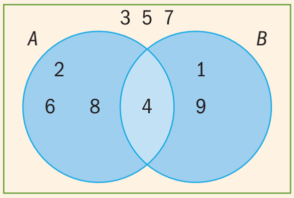
c Explain whether or not the events are independent.

d Verify your answer to **c** using the multiplication rule.

e Verify that $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$. (The addition rule)

Continued on next page

a



b $\mathrm{P}(A) = \frac{n(A)}{n(U)} = \frac{4}{9}$

$\mathrm{P}(B) = \frac{n(B)}{n(U)} = \frac{3}{9} = \frac{1}{3}$

$\mathrm{P}(A \cap B) = \frac{n(A \cap B)}{n(U)} = \frac{1}{9}$

c The events are not independent because whether or not the ticket is an even number affects the probability of it being a square number.

d $\frac{1}{9} \neq \frac{4}{9} \times \frac{3}{9}$

$\mathrm{P}(A \cap B) \neq \mathrm{P}(A) \times \mathrm{P}(B)$

If two events are independent then $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$.

e $\mathrm{P}(A \cup B) = \frac{6}{9} = \frac{2}{3}$

$\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = \frac{4}{9} + \frac{3}{9} - \frac{1}{9} = \frac{6}{9} = \frac{2}{3}$

Verify that $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$.

Practice 1

1 A fair six-sided die has the digits 1, 2, 3, 4, 5, 6 on its faces.

Number on four-sided die						
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	1	2	3	4	5	6

Number on six-sided die

A fair four-sided die has the digits 1, 2, 3, 4 on its faces.

The two dice are rolled simultaneously.

The diagram represents the sample space of the possible outcomes.

a Let A be the event ‘rolling a 1 on the four-sided die’, and B be the event ‘rolling a 1 on the six-sided die’.

Find $\mathrm{P}(A \cap B)$, the probability that you roll a 1 on the four-sided die and a 1 on the six-sided die.

b Explain whether or not the events are independent.

c Verify your result using the multiplication rule.

2 A red die has faces numbered 1, 2, 3, 4, 5, 6 and a green die has faces numbered 0, 0, 1, 1, 2 and 2.

Let A be the event ‘rolling a 2 on the red die’.

Let B be the event ‘rolling a 2 on the green die’.

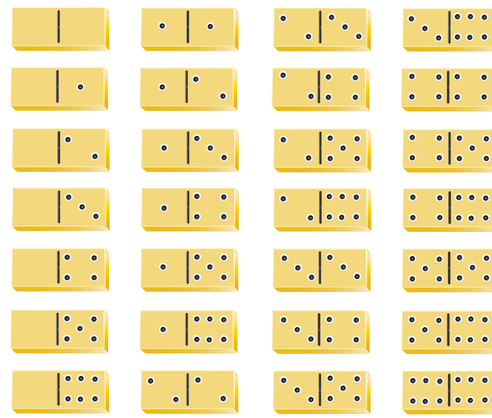
The two dice are rolled together.

a Calculate the probability of rolling a 2 on both the red die and the green die.

b Explain whether or not these events are independent.

c Verify your result using the multiplication rule.

3 Here is a standard set of dominos.



One domino is selected at random. It is then replaced and a second domino is drawn at random.

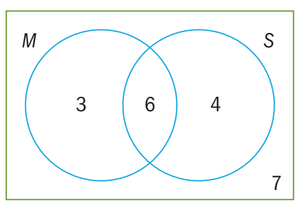
a If S is the set of outcomes of selecting 2 dominos, write down $n(S)$.

b Let A be the event 'at least one of the values on the first domino is a six' and B be the event 'the second domino is a double' (both values are the same). Explain whether or not these events are independent.

c Verify your answer to **b** using the multiplication rule.

Problem solving

4 The Venn diagram shows the number of students in a class taking Mathematics (M) and Science (S). Use it to determine whether or not taking Mathematics and Science are independent events.



5 In a different class, four students take Mathematics only, two take both Mathematics and Science, six take Science only and 12 take neither subject.

Determine whether the choice of taking Mathematics and taking Science are independent events for this class.

6 Students in an after school activity programme register for either trampolining or table tennis. The table shows students' choices by gender:

	Trampolining	Table tennis	Total
Male	39	16	55
Female	21	14	35

A student is selected at random from the group. Find:

a P(male trampoliner)

b P(trampoliner)

c P(female)

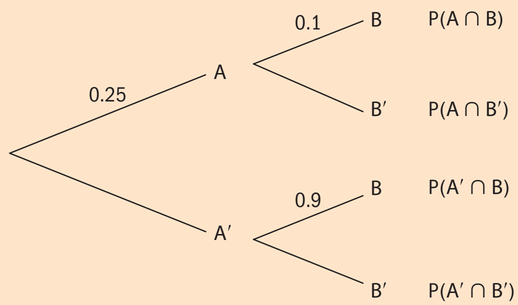
d Determine whether or not the events trampolining and table tennis are independent events.

Exploration 2

The probability that it will rain tomorrow is 0.25. If it rains tomorrow, the probability that Amanda plays tennis is 0.1. If it doesn't rain tomorrow, the probability that she plays tennis is 0.9.

Let A be the event 'rains tomorrow' and B be the event 'plays tennis'.

1 Copy and complete this tree diagram for the events A and B.



2 State whether A and B are independent events.

3 Find the probability that:

a Amanda plays tennis tomorrow, given that it will be raining

b Amanda does not play tennis tomorrow, given that it will be raining

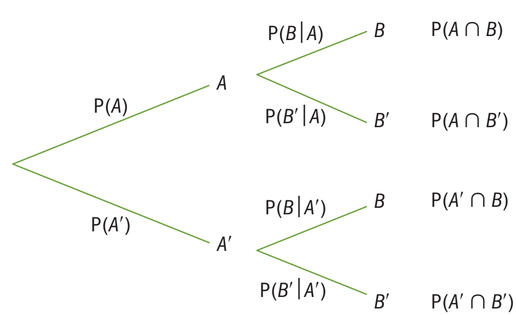
c Amanda plays tennis tomorrow, given that it will not be raining

d Amanda does not play tennis tomorrow, given that it will not be raining.

In step **3** of Exploration 3 you should have found that the probability of Amanda playing tennis is conditional on whether or not it is raining.

In mathematical notation, $P(B|A)$ is 'the probability of B occurring **given that** the condition A has occurred', or 'the probability of B given A'.

Labelling the conditional probabilities on the tree diagram looks like this:



Multiplying along the branches for A and B gives: $P(A \cap B) = P(A) \times P(B|A)$.

ATL

This rearranges to: $P(B|A) = \frac{P(A \cap B)}{P(A)}$

The conditional probability rule:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

You can use this formula to calculate conditional probabilities.

Recall the following rules:

Theorem 1 The addition rule is:
For any events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Theorem 2 The multiplication rule is:
For independent events A and B, $P(A \cap B) = P(A) \times P(B)$.

You can use Theorem 2 as a test for independent events.

If A and B are independent events, then $P(A \cap B) = P(A) \times P(B)$.

A and B are independent events if the outcome of one does not affect the outcome of the other. Writing this using probability notation gives another theorem in the probability axiomatic system.

Theorem 3

If A and B are independent events then $P(B|A) = P(B|A') = P(B)$.

Example 2

The probability that a particular train is late is $\frac{1}{4}$. If the train is late, the probability that Minnie misses her dentist appointment is $\frac{3}{5}$. If the train is not late, the probability that she misses her dentist appointment is $\frac{1}{5}$.
Let A be the probability that the train is late and let B be the probability that Minnie misses her dentist appointment.

a Draw a tree diagram to represent this situation.

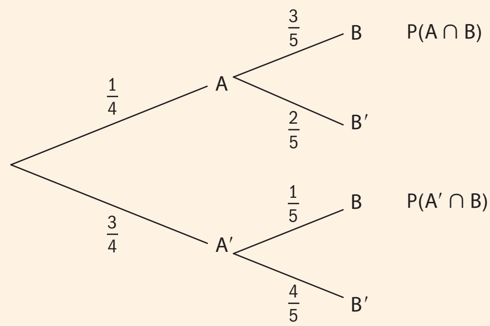
b Find $P(A \cap B)$.

c Find $P(B)$.

d Show that the train being late and Minnie missing her dentist appointment are not independent events using:
i Theorem 2
ii Theorem 3.

Continued on next page

a



B (missing the appointment) is conditional on *A* (train being late).

b $\mathrm{P}(A \cap B) = \frac{1}{4} \times \frac{3}{5} = \frac{3}{20}$

P (*A* and *B*) = P (*A* ∩ *B*).

c $\mathrm{P}(A \cap B) = \frac{3}{20}$, $\mathrm{P}(A' \cap B) = \frac{3}{4} \times \frac{1}{5} = \frac{3}{20}$

$\mathrm{P}(B) = \frac{3}{20} + \frac{3}{20} = \frac{6}{20} = \frac{3}{10}$

B occurs either by P (*A* ∩ *B*) or P (*A*′ ∩ *B*). These two events are mutually exclusive, so add the probabilities.

d i $\mathrm{P}(A) \times \mathrm{P}(B) = \frac{1}{4} \times \frac{6}{20}$

$= \frac{6}{80} = \frac{3}{40}$

$\frac{3}{40} \neq \frac{3}{20}$

So $\mathrm{P}(A) \times \mathrm{P}(B) \neq \mathrm{P}(A \cap B)$ and events *A* and *B* are not independent.

Use Theorem 2.

d ii $\mathrm{P}(B|A) = \frac{3}{5}$

$\mathrm{P}(B|A') = \frac{1}{5}$

$\mathrm{P}(B) = \frac{6}{20} = \frac{3}{10}$

$\mathrm{P}(B|A) \neq \mathrm{P}(B|A') \neq \mathrm{P}(B)$ and therefore events *A* and *B* are not independent.

Use Theorem 3.

In Example 2 the probability of *B* differs depending on whether or not *A* has happened. So the outcome of *A* affects the outcome of *B* and thus the events *A* and *B* are not independent.

Reflect and discuss 1

- Describe another type of problem in probability where events are not independent. Explain how you know they are not independent.
- How is the tree diagram for the probability problem you thought of different from the one in Example 2?

Practice 2

For questions **1**–**4**, draw a tree diagram and determine whether or not events A and B are independent by using either Theorem 2 or Theorem 3.

1 One coin is flipped and two dice are thrown. Let A be the event 'coin lands tails side up' and let B be the event 'throwing two sixes'.

2 Jar X contains 3 brown shoelaces and 2 black shoelaces. Jar Y contains 5 brown shoelaces and 4 black shoelaces. Norina takes a shoelace from Jar X and a shoelace from Jar Y. Let A be the event 'taking a brown shoelace from Jar X' and let B be the event 'taking a brown shoelace from jar Y'.

3 There are 10 doughnuts in a bag: 5 toasted coconut and 5 Boston creme. Floris takes one doughnut and eats it, then he takes a second doughnut.

Let A be the event 'the first doughnut is toasted coconut' and let B be the event 'the second doughnut is toasted coconut'.

4 The probability that Mickey plays an online room escape game given that it is Saturday is 60%. On any other day of the week, the probability is 50%.

Let B be the event 'playing a room escape game' and let A be the event 'the day is Saturday'.

In questions **5**–**8**, determine whether or not the two events are independent. Justify your answer.

5 There are 10 students in your class: 6 girls and 4 boys. Your teacher selects two students at random after putting everyone's name in a hat. When the first person's name is selected, it is not put back in the hat. Let A be the event that the first person selected is a girl. Let B be the event that the second person selected is a girl.

6 Tokens numbered 1 to 12 are placed in a box and two are drawn at random. After the first token is drawn, it is not put back in the box, and a second token is selected. Let A be the event that the first token drawn is number 4. Let B be the event that the second number selected is an even number.

7 According to statistics, 60% of boys eat a healthy breakfast. If a boy eats a healthy breakfast, the likelihood that he will exercise that day is 90%. If he doesn't eat a healthy breakfast, the likelihood that he exercises falls to 65%. Let A be the event 'a boy eats a healthy breakfast'. Let B be the event 'exercises'.

Problem solving

8 Blood is classified in a variety of ways. It may or may not contain B antibodies.

Its rhesus status can be positive (Rh+) or negative (Rh−).

The likelihood of someone having B antibodies is 85%.

The likelihood of being Rh+ is 80%.

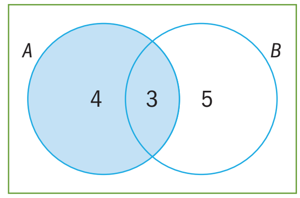
The likelihood of being B+ (having B antibodies and being Rh+) is 68%.

Let X be the event 'having B antibodies'. Let Y be the event 'being Rh+'.

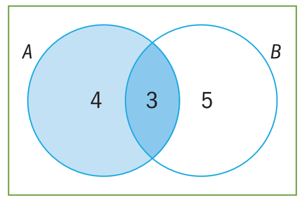
As you ramble on through life, my friend,
Whatever be your goal,
Keep your eye upon the doughnut,
And not upon the hole.
– *The modified Optimist's Creed*

The Venn diagram represents two events, A and B. Find the probability $P(B|A)$.

In this scenario, it is taken that A has already occurred. The sample space is reduced to the elements of A. This is represented in the Venn diagram at the right by the shading on event A.



The region of B that is also in A, or $A \cap B$, represents 'B given A'. Here, the darker shading represents this region.



$$P(B|A) = \frac{\text{number of elements in } A \cap B}{\text{number of elements in } A} = \frac{n(A \cap B)}{n(A)} = \frac{3}{7}$$

In a Venn diagram you can calculate conditional probability by reducing the sample space to the **given** condition.

Example 3

In a group of 30 girls, 12 girls like milk chocolate, 15 like dark chocolate, 6 do not like chocolate. One girl is selected at random.

a Draw a Venn diagram to find the probability the girl likes chocolate.

b Given that she likes chocolate, find the probability she likes milk chocolate.

c Find the probability that she likes dark chocolate, given that she likes milk chocolate.

d Test whether liking milk and dark chocolate are independent events by using:

i Theorem 2

ii Theorem 3.

Let x be girls who like both milk and dark chocolate.

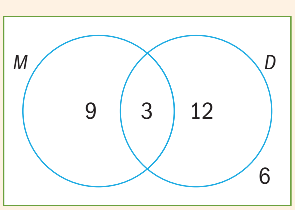
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Draw a Venn diagram. Use Theorem 1 to find $n(A \cap B)$.

$$24 = 12 + 15 - x$$

$$24 = 27 - x$$

$$x = 3$$



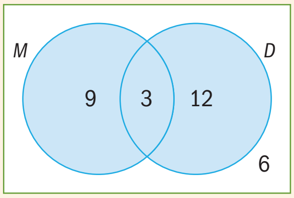
D is the event 'likes dark chocolate'. M is the event 'likes milk chocolate'.

Continued on next page

a $\text{P(likes chocolate)} = \frac{24}{30} = \frac{4}{5}$

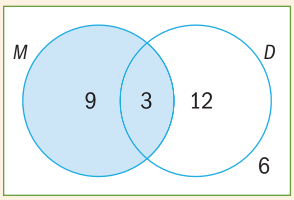
b $\text{P}(M\,|\,\text{likes chocolate}) = \frac{12}{24} = \frac{1}{2}$

Reduce the sample space to only the **given** condition (likes chocolate). Ignore the 6 people who do not like chocolate.



c $\text{P}(D\,|\,M) = \frac{3}{12} = \frac{1}{4}$

Reduce the sample space to *M*.



d **i** M and D are independent events if and only if:

Using Theorem 2.

$\text{P}(M) \times \text{P}(D) = \text{P}(M \cap D)$

$\text{P}(M) = \frac{12}{30} = \frac{2}{5},\ \text{P}(D) = \frac{15}{30} = \frac{1}{2},\ \text{P}(M \cap D) = \frac{3}{30} = \frac{1}{10}$

$\text{P}(M) \times \text{P}(D) = \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}$

$\text{P}(M) \times \text{P}(D) \neq \text{P}(M \cap D)$, therefore they are not independent events.

ii M and D are independent events if and only if:

Using Theorem 3.

$\text{P}(D\,|\,M) = \text{P}(D\,|\,M') = \text{P}(D)$

$\text{P}(D\,|\,M) = \frac{1}{4},\ \text{P}(D\,|\,M') = \frac{2}{3},\ \text{P}(D) = \frac{1}{2}$

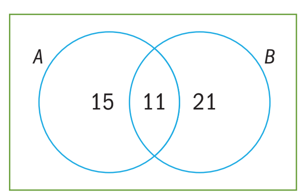
$\text{P}(D\,|\,M) \neq \text{P}(D\,|\,M') \neq \text{P}(D)$, therefore they are not independent events.

ATL

Practice 3

1 From the Venn diagram at the right, find:

a $\text{P}(B\,|\,A)$ **b** $\text{P}(A\,|\,B)$



2 On a field trip, 50 students choose to do kayaking, caving, neither activity, or both.

32 students chose kayaking, 26 students chose caving, and 4 students chose neither activity.

a Represent this information on a Venn diagram.

b Find the probability that a student picked at random:

i chose kayaking

ii chose caving, given that they chose kayaking

iii chose caving

iv chose kayaking, given that they chose caving.

3 In a group of 30 students, 12 students study Art, 15 study Drama, and 8 study neither subject.

a Represent this information on a Venn diagram.

b Find the probability that a student picked at random:

i studies Art

ii studies Drama, given that the student studies Art

iii studies at least one of Drama or Art

iv studies **both** Drama and Art, given the student studies **at least one of** Drama or Art.

4 A group of 25 students are in a show. Of these, 10 of the students sing, 6 sing and dance, 5 students do not sing or dance. A student is selected at random. Determine the probability that, in the show, the student:

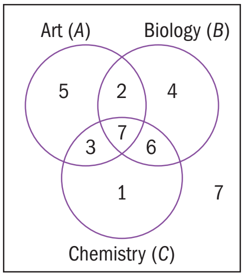
a does not sing

b does not dance

c sings, given that the student does not dance.

Problem solving

5 The Venn diagram shows 35 students' homework subjects.



a Find the probability that a student selected at random had:

i Art homework

ii Biology homework

iii Chemistry homework.

b Find:

i $P(A|B)$ **ii** $P(B|C)$ **iii** $P(C|A')$

c Determine the probability that a student picked at random had homework in all three subjects, given that the student had homework.

Exploration 3

In the first semi-final of a European Cup tournament, Italy play Spain. 42 students were asked to predict the winner.

- 10 males predicted Italy
- 6 males predicted Spain
- 15 females predicted Italy
- 11 females predicted Spain

1 Represent this information in a two-way table.

2 A student was selected at random. Find the probability that this student was male.

Continued on next page

3 By altering the sample space as you did in the Venn diagrams, find the probability the student predicted Italy, given that the student was male.

4 Find the probability that the student was female given that the student predicted Spain.

5 Find:

a $P(M)$ **b** $P(F)$ **c** $P(S)$

d $P(I)$ **e** $P(F|S)$ **f** $P(S|F)$

g $P(M|S)$ **h** $P(S|M)$ **i** $P(F|I)$

j $P(I|F)$ **k** $P(M|I)$ **l** $P(I|M)$

M = Male

F = Female

I = Italy

S = Spain

6 In the other semi-final, England play Germany. The two-way table shows how 42 students predicted the winner.

	England (E)	Germany (G)	Total
Male (M)	6	8	14
Female (F)	12	16	28
Total	18	24	42

a Find the probability that a student selected at random is male.

b Find: **i** $P(M|E)$ **ii** $P(M|G)$

Reflect and discuss 2

Compare the probabilities for the two semi-finals in Exploration 3. Hence make a statement about which events, if any, are dependent and which are independent.

Practice 4

1 There are 28 participants in a golf competition. Of these, 5 are professional male players and 12 are professional female players. There are a total of 18 females playing.

a Use this information to complete the two-way table.

	Male (M)	Female (F)	Total
Professional (P)			
Amateur (A)			
Total			

b Use the table to calculate the following probabilities:

i $P(M|P)$ **ii** $P(M|A)$ **iii** $P(M)$

c Hence, using a theorem, determine whether being male and being a professional are independent events.

2 100 students were asked whether they were left- or right-handed, and whether or not they were on the school soccer team.

Use the results below to complete a two-way table and decide whether or not the events "being left-handed" and "being on the soccer team" are independent.

- P(left-handed) = $\frac{4}{10}$
- P(left-handed and not on the soccer team) = $\frac{3}{10}$
- P(right-handed and not on the soccer team) = $\frac{9}{20}$

Problem solving

3 From the table, determine if being a science teacher is independent of gender (male or female) in the school where this data came from. Justify your answer.

	Science teacher	Other teacher
Male	8	25
Female	16	50

C Developing an axiomatic system

- How can an axiomatic system be developed?
- How does a system facilitate the solving of problems?

Here are three important set definitions that have been previously defined.

1 The universal set U contains all sets under discussion.

For any subset $A \subset U$:

$A \cup U = U$ and $A \cap U = A$

2 There is an empty set $\varnothing$.

For any subset $A \subset U$:

$A \cup \varnothing = A$ and $A \cap \varnothing = \varnothing$

3 For any subset $A \subset U$ there exists a unique complementary set A' such that.

$A \cup A' = U$ and $A \cap A' = \varnothing$

Reflect and discuss 3

- Use the set definitions to help you explain why these probability definitions are true.

  $P(A \cup U) = P(U)$ $P(A \cup A') = P(U)$

  $P(A \cap U) = P(A)$ $P(A \cap A') = P(\varnothing)$

  $P(A \cup \varnothing) = P(A)$ $P(U) = 1$

  $P(A \cap \varnothing) = P(\varnothing)$
- Explain what each definition means, in your own words.

Drawing a Venn diagram may help.

Starting from these rules and Axioms 1, 2 and 3, we can deduce new rules, which will hopefully agree with your intuition about probability.

Exploration 4

1 Suppose you roll a fair four-sided die with sides numbered 1-4. Find the probability of:

a rolling a 3

b not rolling a 3

c rolling an even number

d not rolling an even number

e rolling a number less than 3

f not rolling a number less than 3.

2 Suppose there are 5 choices on a multiple choice question, and a single correct answer.

a What is the probability that you get the answer correct by guessing?

b What is the probability of not getting the answer correct by guessing?

c Repeat steps **a** and **b** for a multiple choice question with just 4 answer choices.

d Repeat steps **a** and **b** for a True/False question.

3 How do your answers for P(A) and P(not A) relate to each other where A is any of the events mentioned in 1 or 2?

4 Generalize a rule for calculating the probability of an event's complement P(A') if you know P(A).

A' means the same thing as not A.

If A is an event then $P(A') = 1 - P(A)$.

Proof of $P(A') = 1 - P(A)$:

By Axiom 2, $P(U) = 1$. And by definition, $P(A \cup A') = P(U)$. Therefore, $P(A \cup A') = 1$.

A and A' are mutually exclusive, so $P(A \cup A') = P(A) + P(A')$, by the addition rule. Hence $P(A) + P(A') = 1$.

This rearranges to $P(A') = 1 - P(A)$.

Proof of $P(\varnothing) = 0$:

$U' = \varnothing$	by definition
$P(U) = 1$	Axiom 2
$P(A') = 1 - P(A)$	complementary events
$P(U') = 1 - P(U)$	substituting U for A
$P(\varnothing) = 1 - P(U)$	because $U' = \varnothing$
$P(\varnothing) = 1 - 1$	because $P(U) = 1$
$P(\varnothing) = 0$	

ATL

Exploration 5

1 Draw three Venn diagrams showing overlapping sets A and B and shade them to illustrate these three relationships:

$A = (A \cap B) \cup (A \cap B')$

$B = (B \cap A) \cup (B \cap A')$

$A \cup B = (A \cap B) \cup (A \cap B') \cup (A' \cap B)$

2 Hence complete these probability statements:

$\mathrm{P}(A) = \mathrm{P}((A \cap B) \cup (A \cap B'))$

$\mathrm{P}(B) = \mathrm{P}(\qquad)$

$\mathrm{P}(A \cup B) = \mathrm{P}(\qquad)$

3 From your Venn diagrams you can see that the events $A \cap B$ and $A \cap B'$ are mutually exclusive and hence you can apply Axiom 3. Hence, complete these statements.

$\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B')$

$\mathrm{P}(B) =$

$\mathrm{P}(A \cup B) =$

4 Add together the statements for $\mathrm{P}(A)$ and $\mathrm{P}(B)$.

5 Rearrange to produce the proof of the addition rule.

Definitions and axioms are the building blocks of any mathematical system. An axiom is a statement whose truth is assumed without proof.

Theorems are established and proven using axioms.

A corollary is a theorem that is a direct result of a given theorem. Usually, a theorem is a larger, more important statement, and a corollary is a statement that follows simply from a theorem.

A proposition is any statement whose truth can be ascertained.

Axioms, propositions, corollaries and theorems for Practice 5

Axiom 1: For any event A, $\mathrm{P}(A) \geq 0$

Axiom 2: $\mathrm{P}(U) = 1$

Axiom 3: If $\{A_1, A_2, A_3, \ldots\}$ is a set of mutually exclusive events then:

$\mathrm{P}(A_1 \cup A_2 \cup \ldots) = \mathrm{P}(A_1) + \mathrm{P}(A_2) + \ldots$

Proposition 1 (complementary events): if A is an event then $\mathrm{P}(A') = 1 - \mathrm{P}(A)$

Corollary 1: $\mathrm{P}(\varnothing) = 0$

Corollary 2: $\mathrm{P}(A) \leq 1$

Theorem 1: For any events A and B, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

Theorem 2: For independent events A and B, $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$

Theorem 3: If A and B are independent events then $\mathrm{P}(B \mid A) = \mathrm{P}(B \mid A') = \mathrm{P}(B)$.

Practice 5

1 A group of 50 students were asked if they liked canoeing and camping. There were 12 students who liked canoeing, 42 students who liked camping, and 8 students who liked both.

a Draw a Venn diagram showing this information.

b Determine how many students did not like either recreation.

c Find the probability that a student chosen at random likes canoeing.

d Determine if liking canoeing and liking camping are independent events.

e State the axioms, propositions and corollaries you used in answering each of the questions **a** through **d**.

2 A group of 100 people were asked if they own a pair of sandals and if they own a pair of slippers. Of these, 48 people own slippers 14 people own both, 18 people own neither.

a Draw a Venn diagram to show this information.

b Determine how many people own sandals.

c Find the probability that a person selected at random owns a pair of sandals, given that the person owns a pair of slippers.

d Determine if the events are:

i mutually exclusive

ii independent.

e State the axioms, propositions and corollaries you used in answering each of the questions **a** through **d**.

3 A survey asked if people were right- or left-handed. There were 30 women and 70 men in the survey; 27 of the women were right-handed, 12 of the men were left-handed. Draw a two-way table showing this information.

a Find the probability of being left-handed, given the person is female.

b Determine whether being left- or right-handed are mutually exclusive events.

c Determine whether being left- or right-handed are independent events.

d State the axioms, propositions and corollaries you used in answering each of the questions **a** through **c**.

You have discovered and used the axioms that lead to important probability theorems. How does this system make it easier to answer probability questions?

Example 4

Jenna and Dan are playing volleyball. The probability that Jenna serves the ball to the back of the court is $\frac{1}{5}$.

If Jenna's serve goes to the back of the court, the probability that Dan returns her serve is $\frac{1}{4}$.

If Jenna's serve does not go to the back of the court then the probability that Dan returns it is $\frac{5}{8}$.

The probability that Jenna's serve goes to the back of the court or that Dan returns it is $\frac{7}{10}$.

Let A be the event 'Jenna serves the ball to the back of the court'.

Continued on next page

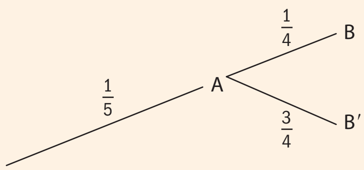
Let B be the event 'Dan returns Jenna's serve'.

a Find $P(A \cap B)$.

b Find $P(B)$.

c Show that A and B are **not** independent.

a



Draw just the part of the tree diagram you need to find $P(A \cap B)$.

$$P(A \cap B) = \frac{1}{5} \times \frac{1}{4} = \frac{1}{20}$$

b $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

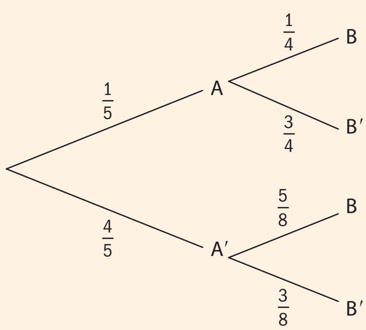
Use Theorem 1.

$$\frac{7}{10} = \frac{1}{5} + P(B) - \frac{1}{20}$$

Use the information given in the question, plus $P(A \cap B)$ which you found from the tree diagram.

$$P(B) = \frac{7}{10} - \frac{1}{5} + \frac{1}{20} = \frac{11}{20}$$

c



Draw a full tree diagram.

In the tree diagram, the probabilities are different for B depending on where the original serve was. Therefore events A and B are not independent.

Objective C: Communicating **v.** organize information using a logical structure
In Practice 6, you will use the axioms of probability to put in place a structure to answer the problems.

Practice 6

1 In a class of 20 students, 12 of them study History, 15 study Geography and 2 students study neither History nor Geography.

Let A be the event 'number of students who study History'.

Let B be the event 'number of students who study Geography'.

a Write down:

i $n(U)$

ii $n(A)$

iii $n(B)$

iv $n(A \cup B)$

v $\mathrm{P}(A)$

vi $\mathrm{P}(B)$

vii $\mathrm{P}(A \cup B)$

b i Let $\mathrm{P}(A \cap B) = x$. Use Theorem 1 to calculate x.

ii Represent the information in a Venn diagram.

c Given that a student picked at random studies History, find the probability that this student:

i also studies Geography

ii does not study Geography.

d Given that a student picked at random does **not** study History, find the probability that this student:

i studies Geography

ii does not study Geography.

e Draw a tree diagram to represent this information.

2 Events A and B are such that $\mathrm{P}(A) = 0.3$ and $\mathrm{P}(B) = 0.4$.

a i Use Axiom 3 to find $\mathrm{P}(A \cup B)$ given that the events are mutually exclusive.

ii Use Theorem 2 to find $\mathrm{P}(A \cap B)$ given that A and B are independent.

b Draw a tree diagram to represent the information in **b ii.**

For **1 a iv**, look at the number of students who study neither History nor Geography and use Proposition 1.

D Making decisions with probability

- What affects the decisions we make?

Objective D: Communication **iii.** move between different forms of mathematical representation
You will need use the table and also a tree diagram in order to solve the problem in Exploration 6.

Exploration 6

A new test can quickly detect kidney disease. Dr Julia Statham performs the test on 140 patients. She knows that 65 of them have kidney disease and 75 of them do not.

A positive test result indicates kidney disease.

A negative test result indicates no kidney disease.

Continued on next page

Here are Dr Statham's results:

	Positive test result	Negative test result	Total
Has kidney disease	30	35	65
Does not have kidney disease	15	60	75
Total	45	95	140

Let T be the event 'patient test result is positive'.

Let D be the event 'patient has kidney disease'.

1 A patient is selected at random. Calculate:

a $P(T)$ **b** $P(T')$ **c** $P(D)$ **d** $P(D')$

The test is successful if it gives a positive result for people with kidney disease and a negative result for people who do not have kidney disease. Dr Statham will use the test if it is 90% accurate.

2 Use the information in the table and your knowledge of conditional probability to decide whether or not the doctor should use the test.

3 Construct a tree diagram that represents the same information as the table.

4 Use the information in your tree diagram to confirm whether or not the doctor should use the test.

5 For this problem, which representation do you prefer: the table or the tree diagram? Explain.

Practice 7

1 After having an operation for an eye condition, a patient lost his sight and claimed that this was caused by the operation. The table shows the hospital's data on treatment for this eye condition.

	Lost sight	Did not lose sight
Patient had the eye operation	25	225
Patient did not have the eye operation	75	675

a Using the information in the table, determine whether the patient's claim is valid. Justify your answer.

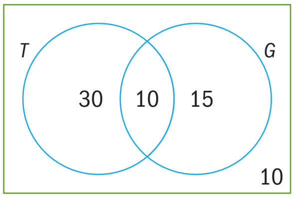
b Represent this information in a tree diagram and use it to determine if the patient's claim is justified.

c Determine which representation (the table or the tree diagram) best illustrates the validity of the patient's claim. Give evidence for your decision.

ATL

2 The Venn diagram represents 65 members of a sports club.

T is the event 'plays tennis' and G is the event 'plays golf'.



a Represent this information in a two-way table.

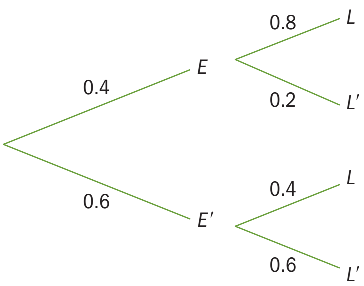
b Use both representations to determine if playing tennis and playing golf are independent events.

c Explain, with reasons, which representation was the easiest to work with.

Problem solving

3 The tree diagram shows information about 100 students at a university.

L is the event 'studies Law' and E is the event 'studies Economics'.



a Use the tree diagram to help you find the following probabilities.

i $P(E)$ ii $P(L|E)$ iii $P(L|E')$ iv $P(E \cap L)$

v $P(E \cap L')$ vi $P(E' \cap L)$ vii $P(E' \cap L')$ viii $P(L)$

b Determine whether studying Law and studying Economics are independent events.

Reflect and discuss 4

- Can you use any probability diagram to represent any situation?
- How do you decide which diagram is most useful for a given situation?

Practice 8

1 The cafeteria sells apples and waffles at morning break. There were 51 students in the cafeteria one morning. Of these, 33 students had apples (A), 18 had waffles (W), and 10 students had neither.

a From the information, and using Theorem 1, draw a Venn diagram.

b Hence, calculate:

i $P(A)$ **ii** $P(W)$ **iii** $P(A|W)$ **iv** $P(W|A)$ **v** $P(W|A')$

c Make a two-way table to represent the information.

d Verify that the results are the same for each form of representation.

e Using Theorem 2 and Theorem 3, test whether having waffles and having apples are independent events.

In questions **2–4**, use the most appropriate representation to solve the problem.

2 70 students who worked in the holidays, 30 of whom are boys, were surveyed about their work. 12 boys and 28 girls worked full-time. The rest worked part-time.

a Given that a student worked part-time, find the probability the student is female.

b Find the probability that a person chosen at random is male and worked full-time.

c Using a theorem as a justification, determine whether 'being male' and 'working part-time' are independent events.

3 The probability of Gregoire arriving at school on time is 60% if there is fog, and 95% if it is clear. The probability of fog is 20%.

a Calculate the probability of Gregoire arriving at school on time.

b Calculate the probability that it is clear, given that Gregoire arrives late.

c Determine whether the events 'getting to school on time' and 'a clear morning' are independent. Justify your answer.

4 Out of 150 students, 30 are left-handed (L) and 80 are male (M). There are 14 left-handed students who are female (F).

a Find $P(F \mid L)$.

b Find $P(L \mid F)$.

c Determine whether the events 'being male' and 'being left-handed' are independent. Justify your answer.

Summary

Theorem 1

IFor any events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

In mathematical notation, $P(B|A)$ is 'the probability of B occurring **given that** the condition A has occurred', or 'the probability of B given A'.

The conditional probability rule:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

You can use **Theorem 2** as a test for independent events. If A and B are independent events, then $P(A \cap B) = P(A) \times P(B)$.

Theorem 3

If A and B are independent events then $P(B|A) = P(B|A') = P(B)$.

If A is an event then $P(A') = 1 - P(A)$.

Mixed practice

1 A lab has 100 blood samples that need to be tested. Of these, 45 are type O, 40 are type A and 15 are type B. Two samples are to be drawn at random without replacement to check for contamination.

a **Draw** a tree diagram to represent this.

b **Find** the probability that the first two samples have the same blood group.

c **Find** the probability that the second sample is type A, given that the first one was type B.

d **Find** the probability that the second sample is not type O, given that the first one was.

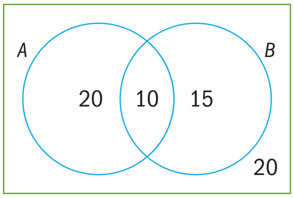
2 From the Venn diagram below, **find**:

a $P(A)$ **b** $P(A \cap B)$

c $P(A|B)$ **d** $P(B|A)$

e $P(B)$ **f** $P(A|B')$

g $P(B|A')$



3 The probability that it will be sunny tomorrow is $\frac{4}{5}$. If it is sunny, the probability that the cafeteria will sell ice creams is $\frac{2}{3}$. If it is not sunny, the probability that the cafeteria will sell ice creams is $\frac{1}{3}$.

a Represent this information in a tree diagram.

b **Find** the probability that it will be sunny and the cafeteria will sell ice creams.

c **Determine** if the sun shining and the selling of ice creams are independent events.

4 S is the event 'Tim is wearing shorts' and R is the event 'it is raining'.

It is known that $P(R) = 0.5$, $P(S|R) = 0.2$, and $P(S|R') = 0.8$.

a Represent this situation in a tree diagram.

b **Use** both Theorem 2 and Theorem 3 to **verify** that events S and T are not independent.

5 Sahar is trying to determine if being tall (here defined as being 180 cm or more) is somehow related to liking basketball. She surveyed 70 students, 30 of which were over 180 cm. 48 students surveyed like basketball, 29 of which were under 180 cm.

a **Draw** a two-way table to represent this.

b **Find** P(tall).

c **Find** P(not tall | likes basketball).

d Are the events 'being tall' and 'liking basketball' independent? **Justify** your answer.

6 In a netball match there are 14 players on the court: 8 have brown hair and 6 have blue eyes. There are 4 students who have neither brown hair nor blue eyes. Let H be the event 'has brown hair' and B be the event 'has blue eyes'.

a **Draw** a Venn diagram to represent this.

b **Determine** these probabilities:

i $P(H)$ **ii** $P(B)$

iii $P(H \cup B)$ **iv** $P(H \cap B)$

v $P(H')$ **vi** $P(B')$

vii $P(H \cup B')$ **viii** $P(H \cap B')$

ix $P(H \cup B)'$

c **Find**:

i $P(H|B)$ **ii** $P(H|B')$

iii $P(B|H)$ **iv** $P(B|H')$

7 A survey was carried out regarding the school dress code. The two-way table shows the results:

	Should have dress code	Should not have dress code	Total
Middle school		30	45
High school	40		
Total	55	110	

a Copy and complete the table.

b **Use** Theorems 2 and 3 to **determine** whether type of school and wanting a dress code are independent events.

Review in context

Identities and relationships

1 The table shows the occurrence of diabetes in 200 people.

	Diabetes	No diabetes
Not overweight	10	90
Overweight	35	65

Let D be the event 'has diabetes'.

Let N be the event 'not overweight'.

a From the table **find**:

i $P(D)$ **ii** $P(N)$

iii $P(D|N)$ **iv** $P(D|N')$

b **Determine** whether having diabetes and not being overweight are independent events.

2 The probability that a randomly selected person has a bone disorder is 0.01. The probability that a test for this condition is positive is 0.98 if the condition is there, and 0.05 if the condition is not there (a false positive).

Let B be the event 'has the bone disorder' and T be the event 'test is positive'.

a **Draw** a tree diagram to represent these probabilities.

b **Calculate** the probability of success for the test. Note the test is successful if it tests positive for people with the disorder and negative for people without the disorder.

3 Michele is making drinks to sell at the school play. She makes a 'Super green smoothie' with baby spinach, cucumber, apple, kiwi and grapes, with a calorific content of 140 kcal per cup, and a 'Triple chocolate milk shake' with chocolate milk, chocolate syrup and chocolate ice-cream, with a calorific content of 370 kcal per cup.

She records the number of MYP and DP students buying each product. MYP students purchase 26 Super green smoothies and 48 Triple chocolate milk shakes. DP students purchase 13 Super green smoothies and 24 Triple chocolate milk shakes.

Determine, using both Theorem 2 and Theorem 3, whether the age of the student (MYP or DP) is independent of their choice of drink. Use a two-way table to answer this question.

4 A survey of 200 000 people looked at the relationship between cigarette smoking and cancer. Of the 125 000 non-smokers in the survey, 981 had cancer at some point in their lifetime. Of the smokers in the survey, there were 1763 people who had cancer.

a **Find** the probability that an individual selected from the study was a smoker who had battled cancer.

b **Find** the probability that someone who had never smoked had cancer.

c Let B be the event 'having cancer' and A be the event 'being a smoker' (includes being a former smoker). **Demonstrate** whether or not these two events are independent. **Justify** your answer mathematically.

d Does your answer in **c** encourage or discourage people from choosing to smoke? **Explain**.

5 The following information was extracted from a skin cancer website.

Statement 1: About 69% of skin cancers are associated with exposure to indoor tanning machines.

Statement 2: In the United States, 3.3 million people a year are treated for skin cancer out of a total population of 330 million.

Statement 3: 10 million US adults use indoor tanning machines.

Let A be the event 'developing skin cancer'.

Let B be the event 'exposure to indoor tanning'.

a **Write down** statements 1, 2 and 3 in probability notation.

b **Determine** whether developing skin cancer and exposure to indoor tanning are independent events.

Reflect and discuss 5

How have you explored the statement of inquiry? Give specific examples.

Statement of inquiry:

Understanding health and validating life-style choices results from using logical representations and systems.